(12) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{r(t) - (n+g) - \rho + n + (1-\theta)g}{\theta}$,

which simplifies to

(13) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{r(t) - \rho - \theta g}{\theta}$.

Equation (13) is identical to the Euler equation in the decentralized equilibrium. See equation (2.20) in the text.

(d) Dividing both sides of equation (9) by $\mu(t)$ leaves us with

(14) $\dfrac{\dot{\mu}(t)}{\mu(t)} = \beta + (n+g) - r(t)$,

where we have substituted for $f'(k(t)) = r(t)$. Note that equation (14) can be written as

(15) $\partial \ln\mu(t)/\partial t = \beta + (n + g) - r(t)$.

Integrating both sides of equation (15) from time $\tau = 0$ to time $\tau = t$ gives us

(16) $\ln\mu(t) - \ln\mu(0) = \left[\beta + (n+g)\right]\tau\Big|_{\tau=0}^{\tau=t} - \int_{\tau=0}^{t} r(\tau)d\tau$.

Using the definition of R(t) and simplifying gives us

(17) $\ln\mu(t) = \ln\mu(0) + \beta t + (n + g)t - R(t)$.

Taking the exponential function of both sides of equation (17) yields

(18) $\mu(t) = \mu(0)e^{\beta t}\, e^{(n+g)t}\, e^{-R(t)}$

Thus $e^{-\beta t}\mu(t)$ is proportional to $e^{-R(t)}\, e^{(n+g)t}$.

This implies that the transversality condition, equation (6), is equivalent to

(19) $\lim_{t\to\infty} e^{-R(t)}e^{(n+g)t}k(t) = 0$.

From equation (2.15) in the text, the household's budget constraint, expressed in terms of limiting behavior, is given by

(20) $\lim_{t\to\infty} e^{-R(t)}e^{(n+g)t}k(t) \geq 0$.

Comparing equations (19) and (20), we can see that the transversality condition will hold if and only if the budget constraint is met with equality. Thus we have shown that the solution to the social planner's problem in the Ramsey model is the same as the decentralized equilibrium. Hence that decentralized equilibrium must be Pareto efficient.

Problem 8.5

The equation of motion for the market value of capital, q, is

(1) $\dot{q}(t) = rq(t) - \pi(K(t))$,

where $\pi'(\bullet) < 0$. The condition required for $\dot{q} = 0$ is given by

(2) $q = \pi(K)/r$.

The equation of motion for capital, K, is

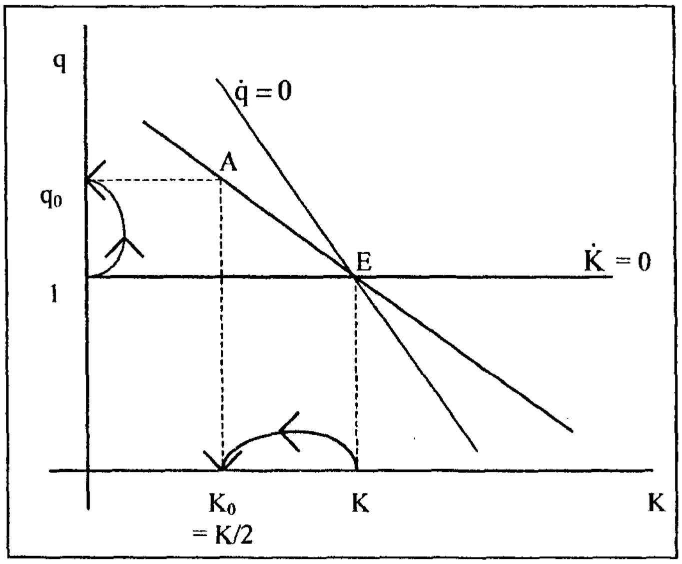
(3) $\dot{K}(t) = f(q(t))$,

where $f(q) = NC'^{-1}(q - 1)$ with $f(1) = 0$ and $f'(\bullet) > 0$. The condition required for $\dot{K} = 0$ is given by

(4) $q = 1$.

(a) The destruction of half of the capital stock does not cause either the $\dot{K} = 0$ or the $\dot{q} = 0$ loci to shift. Both of these are already drawn allowing for K to vary. At the time of the destruction, K falls to $K_0 = K/2$.

For the economy to return to a stable equilibrium, q must adjust so that the economy is on the saddle path. Thus q must jump up to q_0, putting the economy at point A in the figure at right. Intuitively, since profits are higher at the lower K, the capital that is left is more valuable and so the market value of capital is now higher.

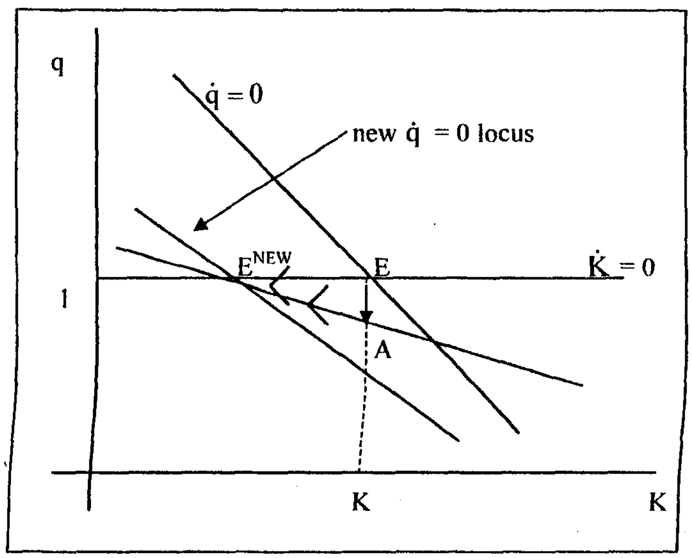


The economy then moves down the saddle path with q falling and K rising. Intuitively, the higher market value of capital attracts investment and so the capital stock begins to build back up. As it does so, profits begin to fall and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is back at its original level. Hence the economy eventually returns to point E.

(b) Profits at a given K are now $(1 - \tau)\pi(K)$ rather than $\pi(K)$. The condition required for $\dot{q} = 0$ is now given by

(5) $q = (1 - \tau)\pi(K)/r$.

At a given K, the value of q that makes $\dot{q} = 0$ is now lower so the new $\dot{q} = 0$ locus lies below the old one. In addition, the slope of the $\dot{q} = 0$ locus is $\partial q/\partial K = (1 - \tau)\pi'(K)/r$ rather than $\pi'(K)/r$. With $(1 - \tau) < 1$, this new slope is less negative and so the $\dot{q} = 0$ locus becomes flatter. The $\dot{K} = 0$ locus is unaffected. See the figure at right.



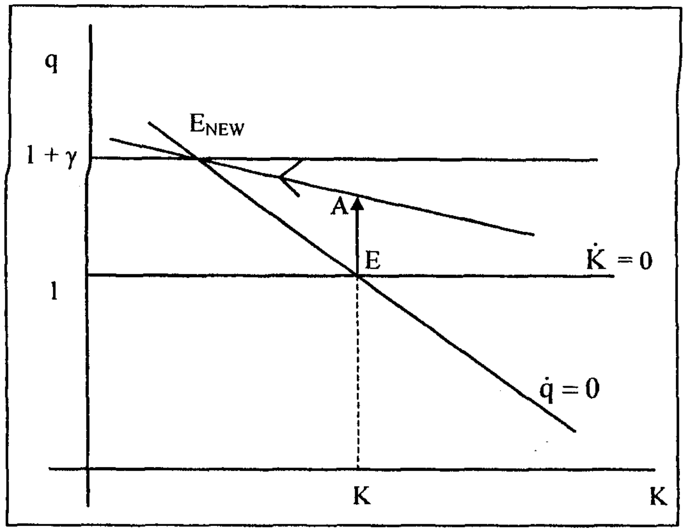
K, the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump down so that the economy is on the new saddle path at point A. Intuitively, since the government is now taking a fraction of profits, existing capital is less valuable and so the market value of capital falls. The economy then moves up the new saddle path with K falling and q rising. Intuitively, the lower market value of capital discourages investment and so the capital stock begins falling. As it does so, profits begin to rise and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is at a permanently lower level. The economy winds up at point E_{NEW} in the diagram. The lower capital and thus higher pretax profits offset the fact that the government takes a fraction of those profits.

(c) One of the conditions required for optimization is that the firm invests to the point at which the cost of acquiring capital equals the value of that capital, q. With this tax on investment, the cost of acquiring a unit of capital is the purchase price (which is fixed at one) plus the tax, γ, plus the marginal adjustment cost, C '(I). Thus analogous to equation (8.21) in the text, we now have

(6) $1 + \gamma + C'(I(t)) = q(t)$.

Since C '(0) is zero, equation (6) implies that I(t) is zero (and thus $\dot{K} = 0$) when $q(t) = 1 + \gamma$. So the equation of the $\dot{K} = 0$ locus is now

(7) $q = 1 + \gamma$.



Thus an investment tax of γ shifts the $\dot{K} = 0$ locus up by γ. The $\dot{q} = 0$ locus is unaffected. See the figure.

K, the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump up so that the economy is on the new saddle path at point A. Intuitively, because the tax will reduce investment, it means that the industry's profits (neglecting the tax) will eventually be higher, and thus that existing capital is more valuable. The economy then moves up the new saddle path until it reaches point E_{NEW}. The capital stock is permanently lower and the pretax market value of capital is equal to $1 + \gamma$; the after-tax market value is again equal to one.

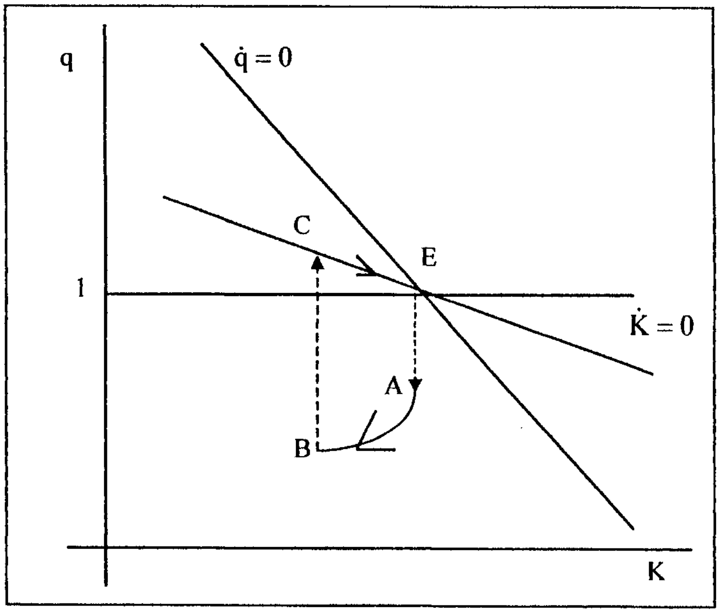
Problem 8.6

The important point is that q is anticipated to jump up discontinuously at the time of the capital levy, time T. Consider what is required, if there is a market for shares in firms, for individuals to be willing to hold those shares through the interval where the one-time tax on capital holdings is imposed. Consider the market value of capital an instant, ε, before the levy and an instant after the levy and then look at what happens as ε goes to zero. The key point is that the market value of capital an instant before the levy, $q(T - \varepsilon)$, must equal (1 - f) times the market value of capital an instant after the levy. If it did not – in light of the levy – holders of shares in firms would be expecting capital losses that they could avoid. Therefore, $q(T - \varepsilon)$ must equal $(1 - f)q(T + \varepsilon)$ or

(1) $q(T - \varepsilon)/q(T + \varepsilon) = (1 - f)$.

For example, if f = 0.10 or ten percent, then the value of q an instant before the levy must equal 90 percent of its value an instant after the levy. Thus at time T, q jumps up to close that 10 percent gap. In addition, that jump must put the economy somewhere on the saddle path in order for the economy to return to a stable equilibrium.

Thus at the time of the news, q must jump down, putting the economy at a point such as A in the figure at right. The economy is then in a region where both q and K are falling. Thus between the time of the news and the time the levy is imposed, the market value of capital and the capital stock are falling. Intuitively, firms begin decumulating capital in anticipation of the one-time levy.



Point A must be chosen so that at the time of the levy, q can jump up by the required amount discussed above and that required jump must put the economy right on the saddle path. The stock of capital does not jump at the time of the levy. Thus at time T, the economy jumps from a point such as B to a point such as C where $q_B / q_C = (1 - f)$.

After the time of the levy, the economy moves down the saddle path, eventually returning to the original equilibrium at point E. Intuitively, once the one-time tax is over with, since K is lower, profits are higher and so investment is attractive once again. Thus the capital stock begins rising back to its initial level.

Problem 8.7

(a) The evolution of the stock of housing is given by

(1) $\dot{H} = I(p_H) - \delta H$.

Thus the condition required for $\dot{H} = 0$ is given by $I(p_H) = \delta H$. That is, in order for the stock of housing to remain constant, new investment in housing – which is an increasing function of the real price of housing – must exactly offset depreciation of the existing housing stock. Differentiating both sides of this expression with respect to H gives us the following slope of the $\dot{H} = 0$ locus:

(2) $I'(p_H) dp_H / dH = \delta$,

or

(3) $dp_H / dH = \delta / I'(p_H) > 0$.

Since $I'(p_H) > 0$, the $\dot{H} = 0$ locus is upward-sloping in (H, p_H) space.

Rental income plus capital gains must equal the exogenous rate of return, r, or

(4) $$\frac{R(H) + \dot{p}_H}{p_H} = r.$$

Solving equation (4) for $\dot{p}_H$ yields

(5) $\dot{p}_H = rp_H - R(H)$.

Therefore the condition required for $\dot{p}_H = 0$ is $rp_H - R(H) = 0$ or $p_H = R(H)/r$. Differentiating both sides of this expression with respect to H gives us the following slope of the $\dot{p}_H = 0$ locus:

(6) $dp_H / dH = R'(H)/r$.

Since $R'(H) < 0$ – rent is a decreasing function of the stock of housing – the $\dot{p}_H = 0$ locus is downward-sloping in (H, p_H) space.